Continuous Projective Measurements - A Non-Hermitian Description

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1. Abstract

The problem of the time of arrival of a quantum system in a specified state is considered in the framework of the repeated measurement protocol and in particular the limit of continuous measurements is studied. It is shown that for a particular choice of system-detector coupling, the Zeno effect is avoided and the system can be described effectively by a non-Hermitian effective Hamiltonian. As a specific example we consider the evolution of a quantum particle on a one-dimensional lattice that is subjected to position measurements at a specific site. By solving the corresponding non-Hermitian wave function evolution equation, we present analytic closed-form results on the survival probability and the first arrival time distribution. Finally we discuss the limit of vanishing lattice spacing and show that this leads to a continuum description where the particle evolves via the free Schrödinger equation with complex Robin boundary conditions at the detector site. Several interesting physical results for this dynamics are presented.

2. Background

The full Hilbert Space

$$\mathcal{H} = \underbrace{\mathcal{S}}_{System} \oplus \underbrace{\mathcal{D}}_{Detector}.$$

$$I = Q + P.$$

Q is projection operator on $\mathcal{S}$ and P is the orthogonal projection on $\mathcal{D}$. Let the system start in the $\mathcal{S}$ and evolve unitarily. At regular intervals of time τ, measure to detect if the system has arrived in $\mathcal{D}$. For a negative measurement, the system (now projected back into $\mathcal{S}$) continues its unitary evolution, until the next measurement and the process is repeated. The experiment stops when we get a positive result indicating arrival into $\mathcal{D}$. The state of the system conditioned on survival (non-detection) after the n-th measurement be denote by $|\psi(n\tau)\rangle$. Then it can be shown [2]

$$|\psi(n\tau)\rangle = \widetilde{U}_\tau^n |\psi(0)\rangle, \quad \widetilde{U}_\tau = QU_\tau Q.$$

The survival probability after n measurements

$$S(n\tau) = \langle\psi(n\tau)|\psi(n\tau)\rangle.$$

We take the Hamiltonian to be

$$H = \underbrace{\sum_{i,j} \gamma_{ij} |i\rangle\langle j|}_{H^{\mathcal{S}}} + \underbrace{\sum_{\alpha,\beta} \gamma_{\alpha\beta} |\alpha\rangle\langle\beta|}_{H^{\mathcal{D}}} + \underbrace{\sum_{i,\alpha}\left[\sqrt{\frac{\gamma_{i\alpha}}{\tau}}|i\rangle\langle\alpha| + \sqrt{\frac{\gamma_{\alpha i}}{\tau}}|\alpha\rangle\langle i|\right]}_{H^{\mathcal{SD}}}$$

where $\{|i\rangle\}$ span $\mathcal{S}$ and $\{|\alpha\rangle\}$ span $\mathcal{D}$. In $\tau \to 0$ limit, the conditioned state $|\psi(t)\rangle$ evolves according to []

$$\imath\frac{\partial|\psi(t)\rangle}{\partial t} = H^{\text{eff}}|\psi(t)\rangle, \quad H^{\text{eff}} = H^{\mathcal{S}} - \imath V^{\mathcal{S}}$$

where

$$V_{ij}^{\mathcal{S}} = \frac{1}{2}\sum_\alpha \sqrt{\gamma_{i\alpha}\gamma_{\alpha j}}.$$

The quantites of interest are the Survival probability $S(t)$ and the First Arrival time distribution $F(t)$ given by

$$S(t) = \langle\psi(t)|\psi(t)\rangle, \quad F(t) = -\frac{dS}{dt} = 2\langle\psi(t)|V^{\mathcal{S}}|\psi(t)\rangle.$$

3. Simple 1-d Hamiltonians

We consider $1-$d lattices with detector at site 0.Introduce the dimensionless parameter α (strength of the detection). The local potential is $2\gamma_0$ except on the site 1 where it is equal to $(\beta+2)\gamma_0$. Hence β is a dimensionless parameter measuring the strength of the potential near the detector. These parameters are capsulated in the complex number $w = \alpha + \imath\beta$.

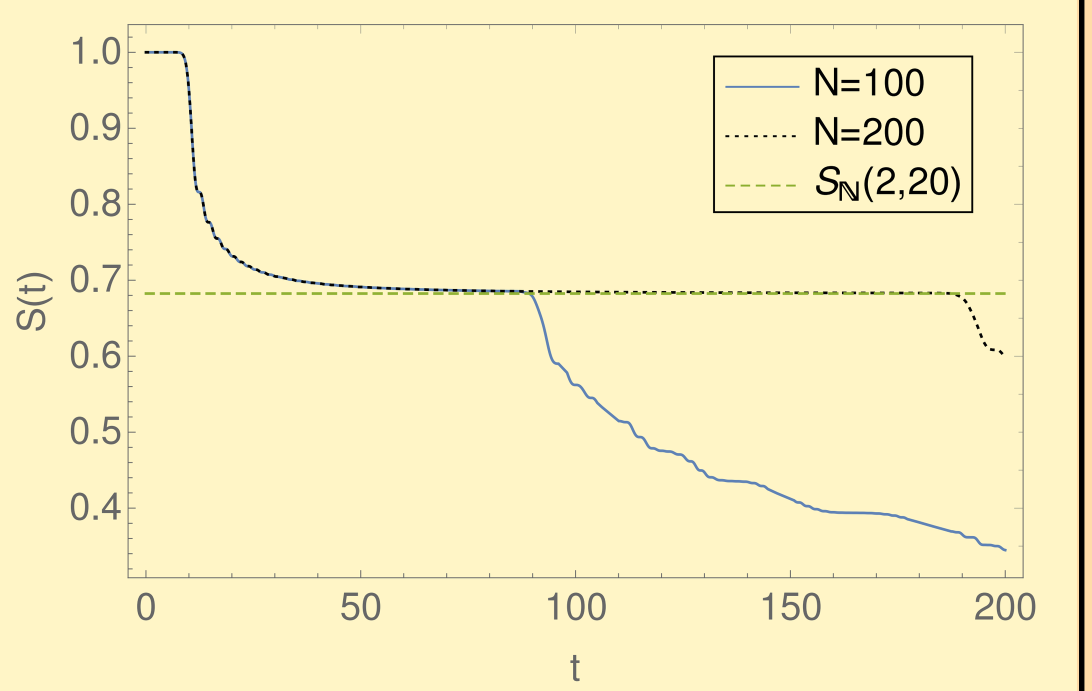
3.1 Finite lattice $\Lambda = \{0,\ldots,N\}$ of size $N \geq 2$

Taking the Hamiltonian to be

$$H = -\gamma_0 \sum_{n=2}^{N}\Big[\,|n\rangle\langle n-1| + |n-1\rangle\langle n| - 2|n\rangle\langle n|\,\Big] + (2+\beta)\gamma_0|1\rangle\langle 1| - \sqrt{\frac{2\alpha\gamma_0}{\tau}}\,(\,|0\rangle\langle 1| + |1\rangle\langle 0|\,)$$

the scaled effective Hamiltonian $H_\Lambda = \dfrac{H^{\text{eff}}}{\gamma_0}$ is

$$H_{\Lambda_N} = -\sum_{n=2}^{N}\Big[|n\rangle\langle n-1| + |n-1\rangle\langle n| - 2|n\rangle\langle n|\Big] + (2-\imath w)|1\rangle\langle 1|.$$



The survival probability $S(t)$ (via numerical simulation) is plotted for lattice sizes $N = 100$ and $N = 200$. In both cases $\psi_i(0) = \delta_{i,20}$, $w = 2$. The dashed line is the value S_∞ obtained from analytical expression in semi-infinite case (See below).

3.2 Semi-infinte Lattice $\Lambda_{\mathbb{N}}$

$$H_{\mathbb{N}} = -\sum_{n=2}^{\infty}\Big[|n\rangle\langle n-1| + |n-1\rangle\langle n| - 2|n\rangle\langle n|\Big] + (2-\imath w)|1\rangle\langle 1|.$$

The Schrödinger equation corresponding to the above Hamiltonian is

$$\imath\frac{\partial\psi_n}{\partial t} = \begin{cases}(2-\imath w)\psi_1 - \psi_2, & n = 1,\\ 2\psi_n - \psi_{n-1} - \psi_{n+1}, & n \geq 2.\end{cases}$$

The analytical form of $S_{\mathbb{N}}(w, n_0)$, where n_0 is the initial position of the particle is shown in [1] to be

$$S_{\mathbb{N}} = 1 - \frac{\Re(w)}{\pi|w|}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \frac{\cos\theta}{\frac{1}{2}\left(|w| + \frac{1}{|w|}\right) + \cos(\theta-\varphi)} - \frac{2\Re(w)}{\pi}\int_0^1 du \frac{u^{2n_0-2}(1-u^2)(1+|w|^2u^2)}{(1+|w|^2u^2)^2 - (2\Im(w)\,u)^2}.$$

4. Continuum Limit of Lattice $\Lambda_{\mathbb{N}}$

It is shown in [1] that the continuum limit of the Schrödinger Equation is

$$\imath\frac{\partial\Psi}{\partial t} = -\frac{\partial^2\Psi}{\partial x^2}, \quad \text{with Robin b.c.} \quad \left[\Psi + \zeta\tfrac{\partial\Psi}{\partial x}\right]_{x=0} = 0,$$

with

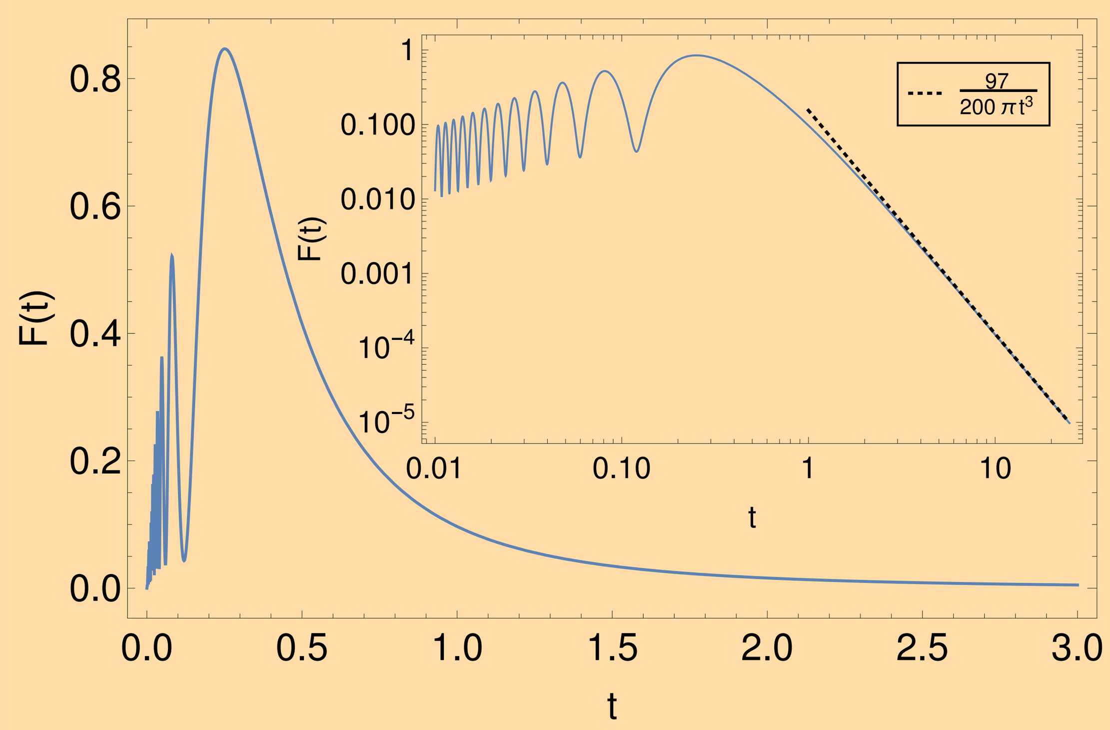
$$\zeta = \lim_{\epsilon\to 0}\epsilon\frac{\imath w}{\imath w - 1}.$$

For an initial state $\Psi_0(x)$, the general solution can be written in terms of the scattering and bound states of the non-Hermitian differential operator in the Schrödinger Equation above. Further

$$\lim_{t\to\infty} t^3 F(t) = -\frac{\Im(\zeta)}{2\pi}|m_{\Psi_0}|^2$$

where

$$m_{\Psi_0} = \int_0^\infty dx\,(x-\zeta)\Psi_0(x).$$



The above graph shows the variation of first passage time distribution $F(t)$ for the $\Psi_0(x) = 1, \quad \text{for } 1 < x < 2$ and 0 elsewhere and $\zeta = 0.2 - 0.5\imath$.

5. Results

- Formulating the problem for general quantum systems with a discrete Hilbert space, we rigorously showed the equivalence between the repeated measurement protocol and the non-Hermitian description.
- For a quantum particle on a 1D lattice with a detector at one site we then solved the corresponding Schrödinger equation with a complex potential.
- We studied the limit of lattice spacing going to 0 to obtain a formulation for the continuum case.

References

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